Exponential Bounds on Graph Enumerations from Vertex Incremental Characterizations

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Graphs

- $G(V, E)$: set of vertices V connected by edges E
- **Simple**: undirected, unlabeled, no self-loops, no multiple edges
- Equivalent graphs determined by isomorphism

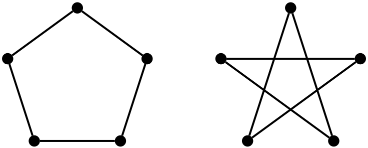


Figure: Isomorphic graphs.

Motivation and Prior Work

- Graph enumeration is a classical problem
- Tree decomposition is a typical approach for exact enumeration
- Focus on distance-hereditary graphs:
  - 1982: Introduction by Cunningham[1]
  - ...: Papers regarding distance-hereditary trees
  - 2009: Approximation of exp growth factor by Nakano *et al.*[2]
  - 2017: Exact enumeration by Chauve *et al.*[3]

[1] Cunningham. 1982.
[2] Nakano, Uehara, and Uno. 2009.
[3] Chauve, Fusy, and Lumbroso. 2017.

Limitations of Chauve *et al.*

- Internal nodes of a split decomposition tree are:
  - star nodes } totally decomposable
  - clique nodes
  - prime nodes } not decomposable
- Distance-hereditary graphs are **totally decomposable**
- Other classes may have prime nodes that are difficult to characterize

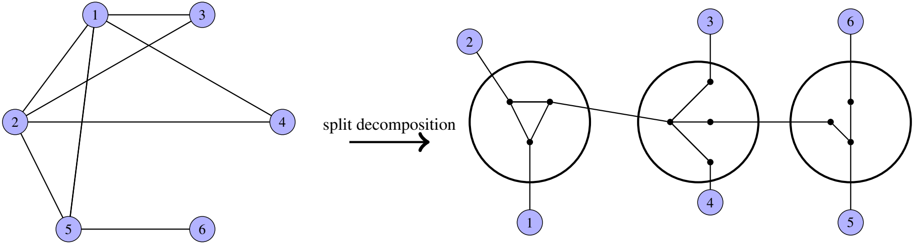


Figure: Graph-labeled tree from split decomposition.

Overview of Nakano *et al.*

- Steps:
  - Constructive view of distance-hereditary graphs (vertex incremental)
  - Expressed this view as **DH-trees**
  - Upper bounded the number of trees with **compact encoding**
- Observation 1: Other graphs can be described using vertex incremental operations
- Observation 2: Analytic combinatorics has more precise tools than compact encoding for bounding trees

Main Idea

- Goal: Combine simplest of both previous results to derive semi-automatic results
- Idea: Sacrifice exactness for an easier methodology
- Demonstrate the following as a general methodology:
  - Define **vertex incremental trees** constructively (focus on surjection)
  - Enumerate using analytic combinatorics
- Proof of concept on two case studies for which we know exact enumerations

Analytic Combinatorics (intuition)

- Describe trees using symbolic rules (grammar)
  - Possible recursive description
  - Translates to generating function, which **gives enumeration**
- Rules:

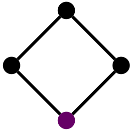
Name	**Symbol**	**Generating Function**
Neutral (element of size 0)	ε	1
Atom (element of size 1)	Z	z
Disjoint Union	$A + B$	$A(z) + B(z)$
Product	$A \times B$	$A(z) \cdot B(z)$
Sequence	$\text{Seq}(A)$	$1/(1 - A(z))$
Set	$\text{Set}(A)$	$\exp(A(z))$

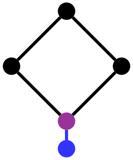
Vertex Incremental Constructions

Vertex Incremental (overview)

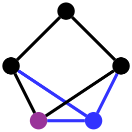
- **Vertex incremental**: necessary and sufficient conditions under which adding a vertex x to a graph of a certain class will produce another graph of that class
- Example:

Starting graph:

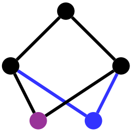




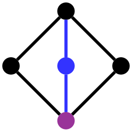
Pendant



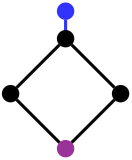
Strong Twin



Weak Twin



Strong Anti-Twin



Weak Anti-Twin

Vertex Incremental (descriptions of graph classes)

Graph Classes	Pendant	Strong twin	Weak twin	Strong anti-twin	Weak anti-twin	Bipartite
3-leaf[4]	1	2				
Cograph[5]		X	X			
Distance-hereditary[6]	X	X	X			
Switch cograph[7]		X	X	X	X	
(6, 2)-chordal bipartite[8]	X		X			
Parity[7]		X	X			X

[4] Gioan and Paul. 2012.
[5] Nakano, Uehara, and Uno. 2009.
[6] Bandelt and Mulder. 1986.
[7] Montgolfier and Rao. 2005.
[8] Cicerone and Di Stefano. 1999.

Vertex Incremental Trees

- **Vertex incremental trees**: Rooted, ordered tree, where internal nodes are labeled with VI ops and leaves are unlabeled
  - Corresponding graph: leaf nodes in bijection with vertices, and internal nodes represent operations used

Vertex Incremental Trees (construction)

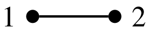
- Start

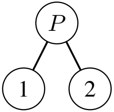
- Add 2 as a pendant of 1

- Add 3 as a strong twin of 2

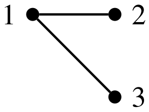
Vertex Incremental Trees (side note)

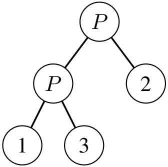
- Add 2 as a pendant of 1



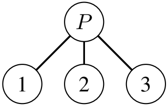


- Add 3 as a pendant of 1





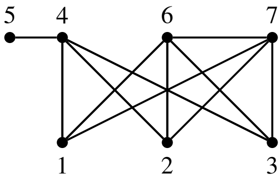
This is equivalently given as:

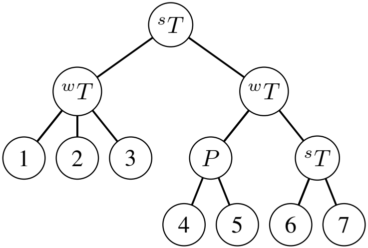


Case 1: Distance-Hereditary Graphs

Distance-Hereditary Graphs

- **Distance-hereditary graph**: graph in which every induced path is the shortest path
- Operations:[9]
  - ${}^{s}T$: strong twin
  - ${}^{w}T$: weak twin
  - P: pendant

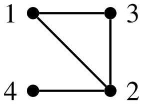


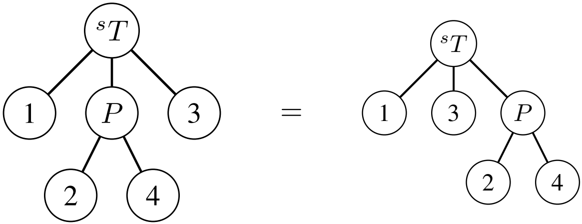


[9] Bandelt and Mulder. 1986.

Normalizing Vertex Incremental Trees

- Multiple vertex incremental trees may correspond to the same graph:
  - Add 2, 3 as strong twins of 1, or
    Add 3, 2 as strong twins of 1
  - Add 4 as a pendant of 2





Normalization Rules

- DH-1. **Commutativity of twins**. The children of a node labeled ${}^wT$ or ${}^sT$ are unordered.
- DH-2. **Commutativity of pendants**. The non-leftmost children of a node labeled P are unordered.
- DH-3. **Connectivity**. The root is not labeled ${}^wT$.
- DH-4. **Associativity of twins**. No child of a node labeled ${}^wT$ can be labeled ${}^wT$, and no child of a node labeled ${}^sT$ can be labeled ${}^sT$.
- DH-5. Any non-leftmost child of a node labeled P cannot labeled ${}^wT$.
- DH-6. If the root has 2 children, it is labeled ${}^sT$.
- DH-7. If the root has 2 children, the labels of the children are either both ${}^wT$ or both P.
- DH-8. **Associativity of pendants. The leftmost child of a node labeled P cannot be labeled P.**

Results: Upper Bound Grammar

$$\mathcal{DH}_T = \mathcal{PR} + \mathcal{SR} + \mathcal{Z}$$

$$\begin{aligned}
\mathcal{PR} &= (\mathcal{S} + \mathcal{W} + \mathcal{Z}) \times \textsc{Set}_{\geq 2}(\mathcal{P} + \mathcal{S} + \mathcal{Z}) \\
\mathcal{SR} &= \textsc{Set}_{\geq 3}(\mathcal{P} + \mathcal{W} + \mathcal{Z}) + \textsc{Set}_{=2}(\mathcal{W}) + \textsc{Set}_{=2}(\mathcal{P}) \\
&\quad + \textsc{Set}_{=2}(\mathcal{Z})
\end{aligned}$$

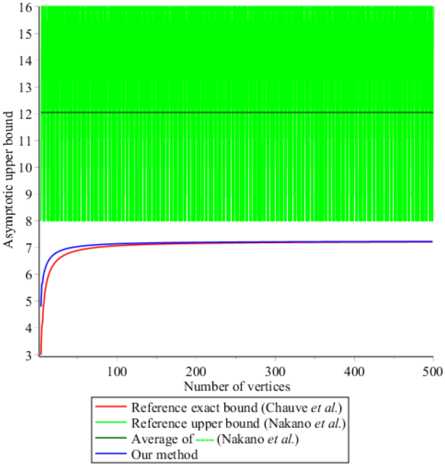
$$\begin{aligned}
\mathcal{P} &= (\mathcal{S} + \mathcal{W} + \mathcal{Z}) \times \textsc{Set}_{\geq 1}(\mathcal{P} + \mathcal{S} + \mathcal{Z}) \\
\mathcal{S} &= \textsc{Set}_{\geq 2}(\mathcal{P} + \mathcal{W} + \mathcal{Z}) \\
\mathcal{W} &= \textsc{Set}_{\geq 2}(\mathcal{P} + \mathcal{S} + \mathcal{Z})
\end{aligned}$$

$$\mathcal{DH}_T = z + z^2 + 2z^3 + 10z^4 + 48z^5 + 270z^6 + \ldots$$

Note: Compared to the actual enumeration, the first few values of this enumeration are indeed an upper bound.

Results: Comparisons

- Our bound: $O\left(7.250^n\right)$
- Reference exact bound: $O\left(7.213^n\right)$
- Reference upper bound: $O\left(12.042^n\right)$



Case 2: Switch Cographs

Switch Cographs and Bicolored Cographs

- **Switch cographs**: graph in which none of its induced subgraphs are C_5, bull, gem, or co-gem graphs
- **Cographs**: graph in which none of its induced subgraphs are P_4
  - **Bicolored cographs**: cograph in which its vertices are colored black or white

Theorem

Let $b = \#\{\text{bicolored cographs on } n-1 \text{ vertices}\}$ and let $s = \#\{\text{switch cographs on } n \text{ vertices}\}$. Then,

$$b \leq s \leq n \cdot b$$

Note: This theorem is derived from previous results by Montgolfier and Rao. Its proof is based on an operation known as the Seidel switch.

Bicolored Cographs

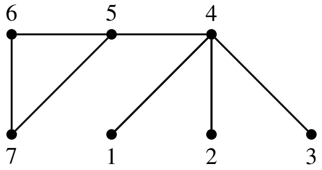
- Exact grammar:

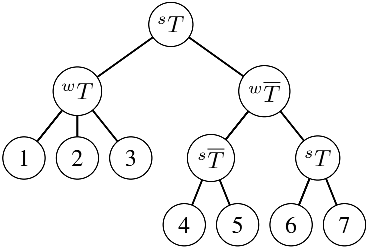
$$\begin{aligned} \mathcal{BC} &= \mathcal{S} + \mathcal{W} + \mathcal{Z} \\ \mathcal{S} &= \textsc{Set}_{\geq 2}(\mathcal{W} + \mathcal{Z}) \\ \mathcal{W} &= \textsc{Set}_{\geq 2}(\mathcal{S} + \mathcal{Z}) \\ \mathcal{Z} &= \mathcal{Z}_{\text{white}} + \mathcal{Z}_{\text{black}} \end{aligned}$$

$$\mathcal{BC} = 2z + 6z^2 + 20z^3 + 80z^4 + 340z^5 + 1570z^6 + \dots$$

Switch Cographs

- Operations:[10]
  - ${}^{s}T$: strong twin
  - ${}^{w}T$: weak twin
  - ${}^{s}\overline{T}$: strong anti-twin
  - ${}^{w}\overline{T}$: weak anti-twin





[10] Montgolfier and Rao. 2005.

Normalization Rules

SC-1. **Commutativity of twins**. The children of a node labeled ${}^{s}T$ or ${}^{w}T$ are unordered.

SC-2. **Commutativity of anti-twins**. The non-leftmost children of a node labeled ${}^{s}\overline{T}$ or ${}^{w}\overline{T}$ are unordered.

SC-3. The non-leftmost children of a node labeled ${}^{s}\overline{T}$ cannot be labeled ${}^{w}T$. The conjugate is also a normalization.

SC-4. The root is not labeled ${}^{s}\overline{T}$ or ${}^{w}\overline{T}$.

SC-5. **Associativity of anti-twins**. The children of a node labeled ${}^{s}\overline{T}$ cannot be labeled ${}^{s}\overline{T}$. The conjugate is also a normalization.

SC-6. The children of a node labeled ${}^{s}\overline{T}$ cannot be labeled ${}^{w}\overline{T}$. The conjugate is also a normalization.

SC-7. **Associativity of twins**. The children of a node labeled ${}^{s}T$ cannot be labeled ${}^{s}T$. The conjugate is also a normalization.

SC-8. **Operator associativity of twins and anti-twins**. The children of a node labeled ${}^{w}T$ cannot be labeled ${}^{s}\overline{T}$. The conjugate is also a normalization.

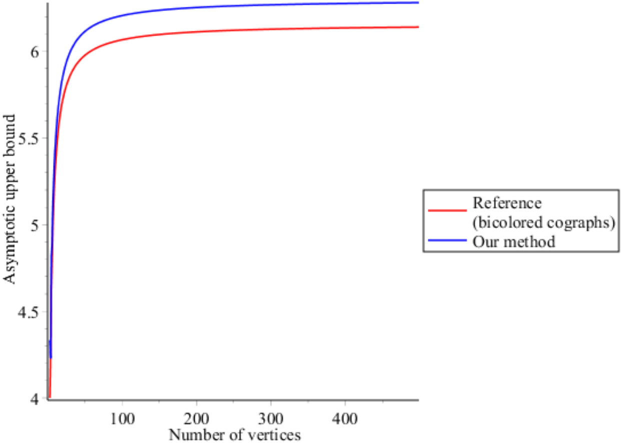
Results: Upper Bound Grammar

$$\begin{aligned}
\mathcal{SC}_T &= \mathcal{ST} + \mathcal{WT} + \mathcal{Z} \\
\mathcal{ST} &= \textsc{Set}_{\geq 2}(\mathcal{WT} + \mathcal{SA} + \mathcal{Z}) \\
\mathcal{WT} &= \textsc{Set}_{\geq 2}(\mathcal{ST} + \mathcal{WA} + \mathcal{Z}) \\
\mathcal{SA} &= (\mathcal{ST} + \mathcal{WT} + \mathcal{Z}) \times \textsc{Set}_{\geq 1}(\mathcal{ST} + \mathcal{Z}) \\
\mathcal{WA} &= (\mathcal{ST} + \mathcal{WT} + \mathcal{Z}) \times \textsc{Set}_{\geq 1}(\mathcal{WT} + \mathcal{Z})
\end{aligned}$$

$$\mathcal{SC}_T = z + 2z^2 + 6z^3 + 26z^4 + 110z^5 + 530z^6 + \ldots$$

Results: Comparisons

- Our bound: $O(6.301^n)$
- Reference exact bound: $O(6.159^n)$



Conclusion

Summary

- Demonstrate that vertex incremental characterizations and analytic combinatorics give asymptotically close upper bound enumerations (as a general methodology)
- Verified upper bounds for distance-hereditary graphs and switch cographs

Next Steps

- Consider other classes of graphs which may be more difficult to construct vertex incremental trees from, e.g., parity graphs
    - Difficulty: bipartite operation is much more difficult to describe
- Compare upper bounds from vertex incremental trees to others from graph-labeled trees/trees derived directly from tree decompositions (e.g., bijoin decomp.)